THE MAXIMAL COVERING LOCATION PROBLEM

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The belief that mathematical location modeling can identify "optimal" location patterns rests on the basis that some realistic objective can be identified and by some measure quantified. For example, in the area of private facilities location analysis, a reasonably accurate statement of the objective of locating warehouses is to minimize the costs of manufacturing and distribution. Since most cost elements included in the objectives of private facility location can be reasonably estimated, the models can picture with some degree of accuracy the real location problem they are designed to solve.

Unlike private facility location analysis, the objectives of public facility location analysis are more difficult to embrace and to quantify. The difficulty in defining direct measures of public objectives has resulted in a search for some surrogate measure with which the decision maker may be comfortable. Two different surrogate measures which have received attention in location models are: (1) total weighted distance or time for travel to the facilities, and (2) the distance or time that the user most distant from a facility would have to travel to reach that facility, that is, the maximal service distance.*

The use of a maximal service distance as a measure of the value of a given locational configuration has been discussed at length by Toregas and ReVelle[1] who show that it is an important surrogate measurement for the value of a given locational configuration. For a given location solution, the maximum distance which any user would have to travel to reach a facility would reflect the worst possible performance of the system. In the regional location of emergency facilities such as fire stations or ambulance dispatching stations, the concept of the maximal service distance is well established.[2,3]

The maximal service distance concept appears in the Location Set Covering

* The concept of a maximal service distance is used throughout this text, although the use of a maximal service time may be equally applicable.

[1] C. Toregas and C. ReVelle, "Optimal Location under Time or Distance Constraints," *Papers of the Regional Science Association*, XXVIII, 1972.

[2] National Board of Fire Underwriters, "Standard Schedule for Grading Cities and Towns of the U.S. with Reference to their Fire Defenses and Physical Conditions," New York, 1956.

[3] H. Huntley, "Emergency Health Services for the Nation," *Public Health Reports*, LXXXV, No. 6 (June, 1970).

Problem.[4,5,6] This problem identifies the minimal number and the location of facilities, which ensures that no demand point will be farther than the maximal service distance from a facility.   By solving the location set covering problem over a range of values of distance, it is possible to develop a cost-effectiveness curve from the pairs of numbers (maximal service distance S, minimum number of facilities to cover).

The number of facilities is used in the location set covering problem as the only cost factor which enters the decision process.   There exists evidence that isolating the number of facilities as the most significant cost parameter may be valid in many real world formulations.   Specifically, if unit costs are independent of site and number of demand points covered, the use of number of facilities is a correct indicator of cost.

Examination of the cost-effectiveness curve reveals that for a given number of facilities there may be *many* location solutions which fulfill the requirement of coverage.   For a desired level of expenditure then (e.g., fixed number of facilities), one may wish to determine the solution with the smallest maximal service distance.[7,8,9]   This point on the curve is identified by Minieka[10] as the left-most corner point of the curve for a stated number of facilities.

It could happen that the distance value obtained in this manner is much larger than the desired distance S.   If it is, the decision maker may shift his attention to concern with the total population covered within S.   Having realized that his resources (facilities) are insufficient to achieve total coverage within his distance goal, the decision maker may seek to cover as many as possible within S using those limited resources.   That is, facing the reality of an insufficient number of facilities, he may relinquish his goal of total coverage within S and attempt instead to locate the facilities in such a way that as few people as possible lie outside the desired service distance.   What is being suggested is a third surrogate for the utility of a location pattern—the population covered within some desirable distance.   The following problem may be stated:

Maximize coverage (population covered) within a desired service distance S by locating a fixed number of facilities.

We will designate this problem as the Maximal Covering Location Problem (MCLP).

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[4] Toregas and ReVelle, *op. cit.*, p. 1.

[5] C. Toregas, C. ReVelle, R. Swain, and L. Bergman, "The Location of Emergency Service Facilities," *Operations Research*, XIX, No. 5 (October, 1971).

[6] C. Toregas and C. ReVelle, "Binary Logic Solutions to a Class of Location Problems," *Geographical Analysis*, V, No. 2 (April, 1973).

[7] S. Hakimi, "Optimum Location of Switching Centers and the Absolute Centers and Medians of a Graph," *Operations Research*, XII (1964), p. 450.

[8] M. Rosenthal and S. Smith, "The M-Center Problem," Operations Research Society of America, 1967 National Meeting, New York, N.Y.

[9] E. Minieka, "The M-Center Problem," *SIAM Review*, XII, No. 1 (January, 1970).

[10] *Ibid.*

A trade-off curve may be developed here as well showing the maximum possible coverage within the desired distance S for various levels of expenditure (that is, for various numbers of facilities). Such a curve can be developed by solving the maximal covering location problem for a fixed S over a range of number of facilities. The corresponding pairs of (number of facilities, maximal population covered) creates the desired curve. For instance, five facilities may be positioned to achieve coverage of 90 percent of the population, but ten facilities may be required for 100 percent coverage. The decision maker may view the required additional funds as money which could be spent in other beneficial ways than coverage within the desirable distance.

Although the decision maker may focus his interest on covering as many as possible within the desired service distance S, he may also be concerned with the quality of service afforded people not covered within that distance. In fact, there may be an undesirable distance to facilities, a distance standard, beyond which no one should be. In short, the decision maker may want to cover the maximum population possible within the desirable distance but ensure at the same time that no one is farther than a distance $T(T > S)$ to his closest facility. Insistence on total coverage within T may be included in order to provide some degree of equity to those not served within the shorter and more desirable service distance S. The closer T can come to the desirable distance S, the fairer the solution is to the people not covered within S.

The following problem may be stated:

Locate a fixed number of facilities in order to maximize the population covered within a service distance S, while maintaining mandatory coverage within a distance of $T(T > S)$.

This problem shall be designated as the maximal covering location problem with mandatory closeness constraints.

MATHEMATICAL FORMULATION OF THE
MAXIMAL COVERING LOCATION PROBLEM

The maximal covering location problem seeks the maximum population which can be served within a stated service distance or time given a limited number of facilities. Defined on a network of nodes and arcs, a mathematical formulation of this problem can be stated as follows:

I. Maximize $\quad z = \sum_{i \in I} a_i y_i$

S.T.

$$\sum_{j \in N_i} x_j \geq y_i \qquad \text{for all } i \in I \tag{1}$$

$$\sum_{j \in J} x_j = P \tag{2}$$

$$x_j = (0, 1) \qquad \text{for all } j \in J \tag{3}$$

$$y_i = (0, 1) \qquad \text{for all } i \in I \tag{4}$$

where

I = denotes the set of demand nodes;

J = denotes the set of facility sites;

S = the distance beyond which a demand point is considered "uncovered" (the value of S can be chosen differently for each demand point if desired);

d_{ij} = the shortest distance from node i to node j;

$x_j = \begin{cases} 1 \text{ if a facility is allocated to site } j \\ 0 \text{ otherwise;} \end{cases}$

$N_i = \{j \in J \,|\, d_{ij} \le S\}$;

a_i = population to be served at demand node i;

p = the number of facilities to be located.

N_i is the set of facility sites eligible to provide "cover" to demand point i. A demand node is "covered" when the closest facility to that node is at a distance less than or equal to S. A demand node is "uncovered" when the closest facility to that node is at a distance greater than S.

The objective is to maximize the number of people served or "covered" within the desired service distance. Constraints of type (1) allow y_i to equal 1 only when one or more facilities are established at sites in the set N_i (that is, one or more facilities are located within S distance units of demand point i). The number of facilities allocated is restricted to equal p in constraint (2). The solution to this problem specifies not only the largest amount of population that can be covered but the p facilities that achieve this maximal coverage.

An equivalent formulation of the maximum covering location problem can be derived by substituting $1 - \bar{y}_i = y_i$ where

$\bar{y}_i = \begin{cases} 1 \text{ if demand node } i \text{ is not covered by a facility within } S \text{ distance} \\ 0 \text{ otherwise.} \end{cases}$

Constraints of type (1) can then be written as

$$\sum_{j \in N_i} x_j \ge 1 - \bar{y}_i \qquad i \in I$$

which is equivalent to

$$\sum_{j \in N_i} x_j + \bar{y}_i \ge 1 \qquad i \in I.$$

This inequality implies that either

$$\sum_{j \in N_i} x_j \ge 1 \qquad \text{or} \qquad \bar{y}_i = 1$$

Stated verbally, either one or more facilities are built within S units of demand point i, or demand i is uncovered and $\bar{y}_i = 1$. After variable substitution, the objective can be written as

$$\text{Maximize} \quad (\sum_{i \in I} a_i + \sum_{i \in I} - a_i \bar{y}_i) .$$

Note that the first sum is a known constant. Since the maximization of a negative quantity is equivalent to a minimization of the positive quantity, the objective function can be simplified to

$$\text{Minimize} \quad \sum_{i \in I} a_i \bar{y}_i \,.$$

This objective can be interpreted as minimizing the number of people that will not be served within the maximal service distance. The complete problem developed by variable substitution is:

II. Minimize $\quad z = \sum_{i \in I} a_i \bar{y}_i$

S.T. $\qquad\qquad \sum_{j \in N_i} x_j + \bar{y}_i \geq 1 \qquad$ for all $i \in I$ $\qquad\qquad\qquad$ (5)

$\qquad\qquad\qquad \sum_{j \in J} x_j = p$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (6)

$\qquad\qquad\qquad x_j = (0, 1) \qquad\qquad$ for all $j \in J$ $\qquad\qquad\qquad$ (7)

$\qquad\qquad\qquad \bar{y}_i = (0, 1) \qquad\qquad$ for all $i \in I$. $\qquad\qquad\qquad$ (8)

This problem seeks to minimize the population left uncovered if p facilities are to be located on a network.

The formulations are equivalent since one can be transformed mathematically into the other by the variable substitution. The latter formulation was utilized in solving this problem with linear programming. The maximal covering location problem has been solved optimally by linear programming and heuristically by several methods.

SOLUTION TECHNIQUES

Heuristic Approaches

The first heuristic considered is called the Greedy Adding (GA) Algorithm. In order to achieve a maximal cover for p facilities under a given service distance, the algorithm starts with an empty solution set and then adds to this set one at a time the best facility sites. The GA algorithm picks for the first facility that site which covers the most of the total population. For the second facility, GA picks the site that covers the most of the population not covered by the first facility. Then, for the third facility, GA picks the site that covers the most of the population not covered by the first and second facilities. This process is continued until either p facilities have been selected or all the population is covered. Details of the algorithm are given in Church.[11]

When solving for the p-facility solution, the GA algorithm automatically calculates maximal cover values for the one- to p-facility problems. It should be noted here than the one-facility maximal covering solution obtained by the GA

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[11] R. Church, "Synthesis of a Class of Public Facilities Location Models," Ph.D. thesis (The Johns Hopkins University, Baltimore, Md., 1974).

algorithm is optimal, for the one-facility solution is by definition the site which covers the most of the total population. However, for solutions where p is greater than 1, optimality is not guaranteed.

It is important to note that the GA algorithm never removes facility sites from the solution set. Therefore, it is possible that a facility site added to the solution set in the early iterations of the algorithm may not be justified later in the algorithm due to subsequent facility site assignments. The presence of a "no longer justified" site in the solution set would imply nonoptimality. The GA algorithm, then, could be improved by including a technique that would reduce the probability of maintaining "no longer justified" sites in the solution set.

The second heuristic which builds on the first is designated as the Greedy Adding with Substitution (GAS) Algorithm. The GAS algorithm determines new facility locations at each iteration just as the Greedy Adding Algorithm did, but in addition, seeks to improve the solution at each iteration by trying to replace each facility one at a time with a facility at another "free" site. If improvement is possible, the facility site chosen to replace a particular facility is the one which gives the greatest improvement in the objective. The GAS algorithm is outlined in Church.[12]

The GAS algorithm, like the GA algorithm, automatically calculates maximal coverage for problems with one, two, ... to p facilities. Again, however, global optimality is not guaranteed.

The method of trying to replace one at a time the facilities in the solution in an effort to improve coverage is similar in concept to the λ-optimum method introduced by Shen Lin in solving the traveling salesman problem.[13] In applying the λ-optimum concept to the maximal covering location problem, one can state that a maximal covering solution is λ-optimal if it is impossible to obtain a solution with greater cover by replacing any λ of its facilities by any other set of λ facility sites. Using this definition, GAS is classified as a 1-optimum algorithm.

Conceptually, the GAS algorithm is similar to the Ignizio heuristic.[14] The difference is that GAS replaces any facility in the solution set by another facility if resultant coverage increases, whereas the Ignizio heuristic replaces only the facility sites which contribute less to the total coverage than the contribution made by the last added facility. The λ-optimum method applied to a random start solution or heuristically derived solution has been used by Ignizio and Harnett to solve the location set covering solution.[15] The Ignizio heuristic has

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[12] *Ibid.*

[13] S. Lin, "Computer Solutions of the Traveling Salesman Problem," *The Bell System Technical Journal*, XLIV (December, 1965).

[14] J. P. Ignizio and R. E. Shannon, "Heuristic Programming Algorithm for Warehouse Location," *AIIE Transactions*, II, No. 4 (December, 1970).

[15] J. P. Ignizio and R. M. Harnett, "Heuristically Aided Set-covering Algorithms," Department of Industrial Engineering, University of Alabama at Huntsville, 1972.

been utilized by Case and White[16] to solve a problem related to the maximal covering problem.

Case and White sought the *maximum number of demand points* rather than the maximal population which could be covered with a stated number of facilities. Their problem is a subproblem to the maximal covering problem with $a_i = 1$ for all demand points. Like the Ignizio heuristic, GAS allocates facilities to sites one at a time in a steepest ascent manner and employs a subroutine which replaces facilities in the solution which are no longer justified, due to subsequent facility assignments. From a more general viewpoint, the GAS and Ignizio heuristics resemble a warehouse location algorithm developed by Kuehn and Hamburger.[17]

Linear Programming

Linear programming may also be applied to the maximal covering location problem and will obtain globally optimal solutions. All that is necessary is to restrict the variables x_j and $\bar{y}_i$ in Problem II to be nonnegative instead of zero or one.

In an optimal linear programming solution to this problem, the variable x_j will never be greater than one (unless total coverage were achieved). If some x_j were greater than one at optimality it could be reduced to one without violating any constraint (5). Further, the reduction could be applied toward increasing some other x_j within the constraint that total facilities equal p (6). This will allow some $\bar{y}_i$ to be decreased leading to a decrease in the objective. This contradicts the assumption of optimality.

Since the objective function keeps each variable $\bar{y}_i$ as small as possible and because it is never necessary for $\bar{y}_i$ to be greater than one in order to satisfy constraints (5), $\bar{y}_i$ will never be greater than one in an optimal linear programming solution. Since x_j will never be greater than one unless total coverage is achieved, the linear program will terminate with an optimal solution where $0 \leq x_j \leq 1$ for all $j \in J$ and $0 \leq \bar{y}_i \leq 1$ for all $i \in I$.

If linear programming is applied and total coverage is not possible, the optimal solution will have either:

Case 1: All $x_j, \bar{y}_i = (0, 1)$, called an "all-integer answer";
Case 2: Some x_j's fractional, called a "fractional answer."

If the linear program terminates with all x_j and $\bar{y}_i = (0, 1)$, the optimal solution to the maximal covering location problem has been determined. If the linear program terminates as Case 2, the optimal solution given is not feasible to the zero-one problem. More work will be necessary to obtain an optimal zero-one

[16] J. White and K. Case, "On Covering Problems and the Central Facilities Location Problem," unpublished paper, Virginia Polytechnic Institute and State University, Blacksburg, Va., 1973.

[17] A. Kuehn and N. Hamburger, "A Heuristic Approach for Locating Warehouses," *Management Science*, X (1963), p. 643.

solution. The solution value does, however, make explicit the lower bound on the objective function.

The Mathematical Programming System (MPS) on the IBM 360 Model 91 computer was used to obtain optimal linear programming solutions to maximal covering problems. The problems were stated as minimizations of population uncovered. In order to solve the problem, a Fortran IV program was developed to write the problem on a disc file. This disc file was developed in such a manner that the MPS system could access the problem file and perform the necessary linear programming algorithm.

When the solution for more than one number of facilities is desired for a given network and a stated service distance S, postoptimal routines of the MPS system were used. For example, suppose that the desired range in the number of facilities was $p = 1$ to $p = 10$. The problem file was set up for the linear program with $p = 1$. After the MPS system obtained an optimal solution for this value of p, the option for parameterization of the right-hand side was used to obtain solutions for $p = 2$ to $p = 10$. The cost of computing each additional solution ($p = 2$ to $p = 10$) is only a small increment over that for the first solution ($p = 1$).

Resolving Fractional Solutions to the Linear Program

Our computational experience is that almost 80 percent of the time the linear programming solution is all zero-one. Only approximately 20 percent of the time it is necessary to resolve fractional problems. Two different techniques, a method of inspection and the method of Branch and Bound, have been used to obtain all integer solutions when the linear program terminates fractional. The first method shows that at times it is possible to determine an all-integer optimal solution to a p-facility problem directly from the fractional p-facility solution and from an optimal all-integer ($p-1$)-facility solution. The explanation of such a possibility and the ease with which the fractional resolution can be made can be best described by an actual example. Consider the following information:

> The maximal covering problem was solved over the range $p = 1$ to $p = 9$ for a 55-node network by the MPS system. The solutions were all zero-one except for the solutions at $p = 8$ and $p = 9$. The total population of nodes on the network is 640. The seven-facility solution covers all but two nodes; namely, demand node A and demand node B. The poulation at A is five and the population at B is two. The smallest population of any node is two. Each demand node is a possible facility site. The population covered by the optimal but fractional linear programming solution with $p = 8$ was 639.

The problem, then, is to determine, if possible, the optimal all-integer solutions for $p = 8$ and $p = 9$ using only the above information. In order to do this the following two observations are important:

1) Since the optimal seven-facility solution covers all but A and B, the seven-

facility solution covers 633 (i.e., 640 − 5 − 2 = 633).

2) Either the optimal all-integer eight-facility solution will totally cover, or given that it does not, it can cover at most all the population except the smallest node population. This means that either the optimal eight-facility solution will cover 640 or at most 638.

Since the optimal $p = 8$ linear program covers 639, the optimal all-integer $p = 8$ solution cannot cover. Thus, observation (2) shows that the all-integer optimal answer can cover at most 638. By adding node A to the seven-facility solution, an eight-facility solution set is formed where coverage is 638. Since this is the best an eight-facility solution can possibly cover, this solution must be optimal. Since the eight-facility solution cannot totally cover, a nine-facility solution that can totally cover would indeed be optimal. Such a solution is created by adding nodes A and B to the seven-facility solution. Thus, the information provided in a $(p − 1)$-facility solution may be useful in determining an all-integer p-facility solution, when such is not provided by linear programming.

The method of Branch and Bound[18] was applied to fractional linear programming solutions which did not yield to the method of inspection. By branching on a fractional variable, i.e., by setting either $x_j = 1$ or $x_j = 0$, two more linear programming problems are created. If the linear programming solution to each of these two problems has all $x_j = (0, 1)$, the solution that has the smaller value of the objective function is optimal. If the solution to the problem with the smaller value of the objective is fractional, additional branching and computations may be required.

PERFORMANCE OF THE ALGORITHMS

Two test networks were used to illustrate the value of the maximal covering location approach as well as to evaluate the performance of the two heuristics and the optimal linear programming approach. The first network is a 30-node problem designed by Rojeski and ReVelle.[19] The second network contains 55 nodes and was described by Swain.[20] Each network ensures randomness and feasibility of the internode distances for geographic interpretation. It is assumed that each demand node is a possible facility site.

The optimal linear programming approach has been applied to many maximal covering location problems defined on the 55-node network used by Swain. In obtaining all-integer optimal solutions, fractional linear programming solutions were resolved by first attempting to use the method of inspection described earlier.

[18] H. M. Wagner, *Principles of Operations Research with Applications to Managerial Decisions* (Englewood Cliffs, N.J.: Prentice-Hall, 1969).

[19] P. Rojeski and C. ReVelle, "Central Facilities Location under an Investment Constraint," *Geographical Analysis*, II, No. 4 (1970).

[20] R. Swain, "A Decomposition Algorithm for a Class of Facility Location Algorithms," Ph.D. thesis (Cornell University, Ithaca, N.Y., 1971).

TABLE 1
COMPARISON OF ALGORITHMS
Problem Data from 55-node Network, Ref. 19

Value of S	Range of p and Number of Solutions	LINEAR PROGRAMMING				GA ALGORITHM		GAS ALGORITHM	
		Fractional Solutions (values of p)	Fractions Resolved by (values of p)	Branch and Bound Needed (values of p)	Total MPS Time (sec) IBM360/91	Non-Optimal Solutions (values of p)	Total Est. Execution Time (sec) (IBM 7094)	Non-Optimal Solutions (values of p)	Total Est. Execution Time (sec) (IBM 7094)
10.00	1 to 9	8, 9	8, 9	—	7.38	4, 5, 6, 7, 8, 9	10.8	4, 5.7, 8	23.6
12.00	1 to 7	3, 4	—	3, 4	29.35*	2, 3, 4, 5, 6	4.2	2, 3, 4, 7 6, 7	18.2
15.00	1 to 5	5	5	—	6.33	2, 3, 4, 5	6.0	3, 4, 5	13.5
19.00	1 to 3	3	—	3	7.79**	2, 3	1.8	2	8.1
20.00	1 to 3	—	—	—	4.25	2	3.6	0	7.8
Total number of Solutions	27	6	3	3	—	18	—	14	—

* Includes 19.18 seconds for four additional solutions obtained by Branch and Bound, none of which preserved a basis set for restart.

** Includes 1.75 seconds for one additional solution obtained by a single branching.

If this was not possible, the method of Branch and Bound was used.

Table 1 compares the performance of the linear programming approach and the heuristics in solving 27 maximal covering location problems defined on the 55-node network. For each service distance S, maximal covering solutions for a range of p values were obtained using the MPS system with the parameterization option. Almost 80 percent of the linear programs terminated all-integer optimal, and nearly 90 percent of the total problems were solved by the combination of linear programming and the method of inspection. Only 10 percent of the time was Branch and Bound necessary to resolve fractional problems. In every case of application of the Branch and Bound algorithm, all that was necessary to obtain an all-integer optimum was the solution of several linear programs. Computation times are given for the MPS execution time only.

The GA algorithm was performed by a Fortran IV program on the IBM 7094 system. Computation time includes the set-up of the particular maximal covering location problem from basic data as well as the execution time of the algorithm. From the table, it can be seen that the GA algorithm is nonoptimal more than 60 percent of the time. The GAS algorithm was also performed by a Fortran IV program on the IBM 7094 system. Computation time includes the development of problem information from basic data as well as execution time of the algorithm. The GAS algorithm was nonoptimal almost 50 percent of the time.

For each heuristic solution generated, the coverage was no lower than 90 percent of the corresponding optimal coverage value. In other words, the heuristics applied to the 55-node network attained close to optimal solutions, if not the optimal solutions. One reason for this good performance is the fact that the 55-node network contains a very dense area that can generally be covered by one facility. This tends to make the first facility's coverage so large that it always remains the most significant portion of any solution. The large significance of the first facility tends to outweigh any differences in coverage due to added facilities which are either heuristically or optimally determined. While it is true that the heuristics have done well in this problem, it is also true that the only reason we can assess their quality is that we have an efficiently derived optimal solution to which they may be compared. Enumeration to determine optimal solutions for comparison to heuristic solutions would generally be out of the question because of the computer time it would consume.

For any desired value of S (maximal service distance) and p value (the number of facilities to be located), a maximal covering location problem can be defined. By holding S fixed, and solving over a range of p values, one can create a cost-effectiveness curve which outlines the change in effectiveness (in terms of population covered) as the system cost is altered by increasing or decreasing the number of available facilities.

Figure 1 shows the cost-effectiveness curve for the maximal covering location problem defined on the 30-node problem with the maximal service distance fixed

at $S = 2.0$. This curve represents optimal maximal covering solutions obtained by the MPS system with the parameterization option. The increase in number of people covered by the addition of one facility decreases as the total number of facilities increases. The optimal solution for $p = 12$ also represents an optimal solution to the location set covering problem defined for $S = 2.0$. This is true, since $p = 12$ represents the lowest p value for which a 100 percent coverage is attainable.
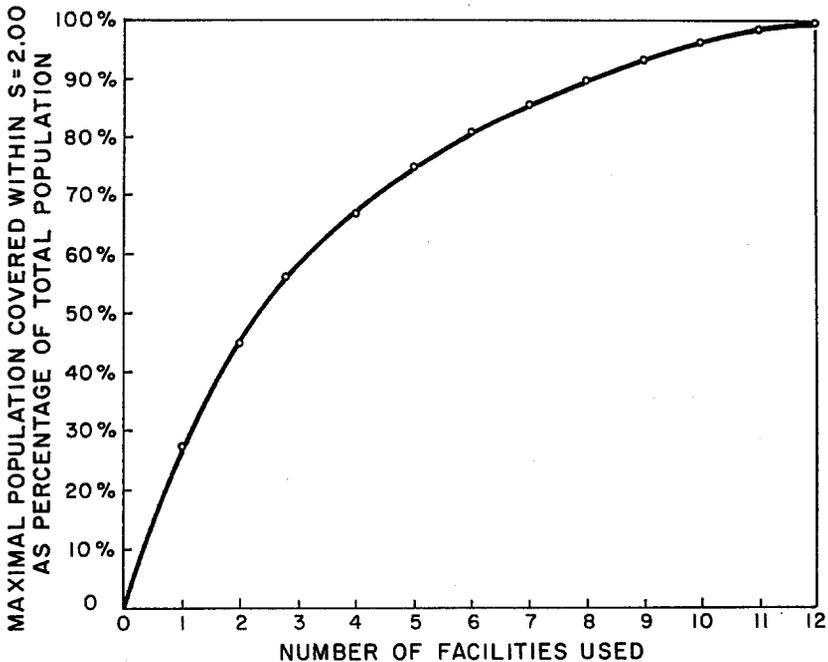

FIGURE 1.   A Cost-effectiveness Curve Derived for $S = 2.00$ (Data for 30-node Problem from Rojeski and ReVelle, Ref. 19)

An important choice for the decision maker is deciding what level of expenditure (how many facilities can be used) can be justified by the resultant coverage. For example, even if there are enough funds to build twelve facilities and cover 100 percent, the eight-facility solution and 90 percent coverage may be very acceptable. Further, the reduced location cost may allow funds to be spent upgrading other levels of service not related to location. Thus, the cost-effectiveness curve yields much valuable information to the decision maker. Such information would seem beneficial to the formulation of logical and sound location plans.

MAXIMAL COVERING WITH
MANDATORY CLOSENESS CONSTRAINTS

Even though a decision maker might seek to maximize coverage within some desirable maximal service distance, there may be an undesirable distance to

facilities beyond which no demand should be. Insistence on some type of mandatory closeness was introduced as a mechanism for achieving some form of fairness for those not served within the shorter and more desirable service distance S.   By including this mandatory closeness concept in the maximal covering location problem, the following location problem was postulated:

> Locate p facilities at possible sites on the network to maximize the population that can be covered within a given service distance S while at the same time ensuring that the users at each point of demand will find a facility no more than T distance $(T > S)$ away.

A mathematical formulation of this problem is very similar to Problem I.   The objective function and the constraint on total number of facilities are identical with those of the earlier formulation.   There are now, however, two types of covering constraints, namely:

$$\sum_{j \in N_i} x_j \geq y_i \qquad \text{for all } i \in I \tag{9}$$

and

$$\sum_{j \in M_i} x_j \geq 1 \qquad \text{for all } i \in I \tag{10}$$

where

$$N_i = \{j \,|\, d_{ij} \leq S\}$$
$$M_i = \{j \,|\, d_{ij} \leq T\}$$

and all other notation is as previously defined.

Note that since $T > S$, M_i contains the set N_i.   This is the maximal covering location problem with mandatory closeness constraints.   The mandatory closeness constraints (10) ensure for each demand point i that there must be at least one facility within the distance T.   Unlike the problem without mandatory closeness, this problem has feasible solutions only for certain integer values of p, the number of facilities.   The lower limit of the range of values of p is the minimal number of facilities necessary to cover all demand points within T distance units.   The same variable substitution may be used to transform this to a problem which minimizes population not covered within S while maintaining complete coverage within T.

Suppose that for a certain problem the smallest number of facilities necessary for total coverage within T is p^*.   A p^*-facility solution that does cover all within T is an optimal location set covering solution since p^* is the smallest number of facilities necessary to cover all within T.   There could well be several different p^*-facility solutions that maintain total coverage of everyone within T distance units.   One could seek to maximize coverage within S (where $S < T$) while maintaining total coverage within T by relocating the p^* facilities.   This implies that we would be picking among the alternative optima to the location set covering problem the solution that achieves the largest coverage within the smaller distance S.   The

use of maximal covering with mandatory closeness constraints, then, can be considered a technique for obtaining more desirable solutions than the location set covering problem. The new solutions will continue to cover the entire population and are desirable in the sense that they provide greater population coverage within an even closer and more desirable distance.

The maximal covering location problem with mandatory closeness constraints has been solved by applying linear programming to the minimizing form of the problem, that is, by minimizing the number not covered within S while maintaining complete coverage within T. The problem was solved using the MPS 360 system on the IBM 360 Model 91 computer. Again, if more than one p-facility solution was desired, the MPS parameterization option was used. Essentially, the problem with mandatory closeness constraints is set up with the number of facilities equal to the smallest number of facilities necessary to cover all demand within T distance units; the parameterization option was used to develop optimal answers for increasing values of p. It is not necessary to allow p to be greater than the number of facilities needed to cover all demand nodes within the smaller distance S.

In limited computational experience, eight out of eleven linear programs terminated all-integer, and hence optimal to the zero-one problem. All fractional linear programming solutions were able to be resolved by the method of inspection. A heuristic approach to solve the problem is also possible.[21]

Suppose that a decision maker is interested in covering all demand within a service distance of 15. Let us say that the 55-node network represents his particular problem. His goal, we will assume, is to achieve complete coverage with the fewest possible facilities. The decision maker is implicitly structuring the location set covering problem[22] for $S = 15$. This is the minimum number of facilities which accomplishes coverage within a specified maximum distance. Figure 2 gives a location set covering solution for a service distance of $S = 15$ on the 55-node network. In this case five facilities are required to cover all nodes within a distance of 15.

The circled nodes on the graph given in Figure 2 represent the five sites chosen to house facilities. The nodes of the graph are partitioned into sets, each node being assigned to its closest facility. In Figure 2 there are two types of partitions: those enclosed by solid lines and those indicated by dashed lines. The solid line partitions indicate which nodes are covered within $S = 15$. The dashed line partitions denote which nodes are covered within $S = 10$. The reason for showing which nodes are covered by this solution within a distance of 10 will become clear momentarily. For example, node 42 is in the solid line partition surrounding a facility located at node 25. This means that the closest facility to node 42 is located at node 25 and that the distance from node 42 to that facility is less than 15. Furthermore, node 42 is not in the dashed line partition for the facility

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[21] Church, *op. cit.*
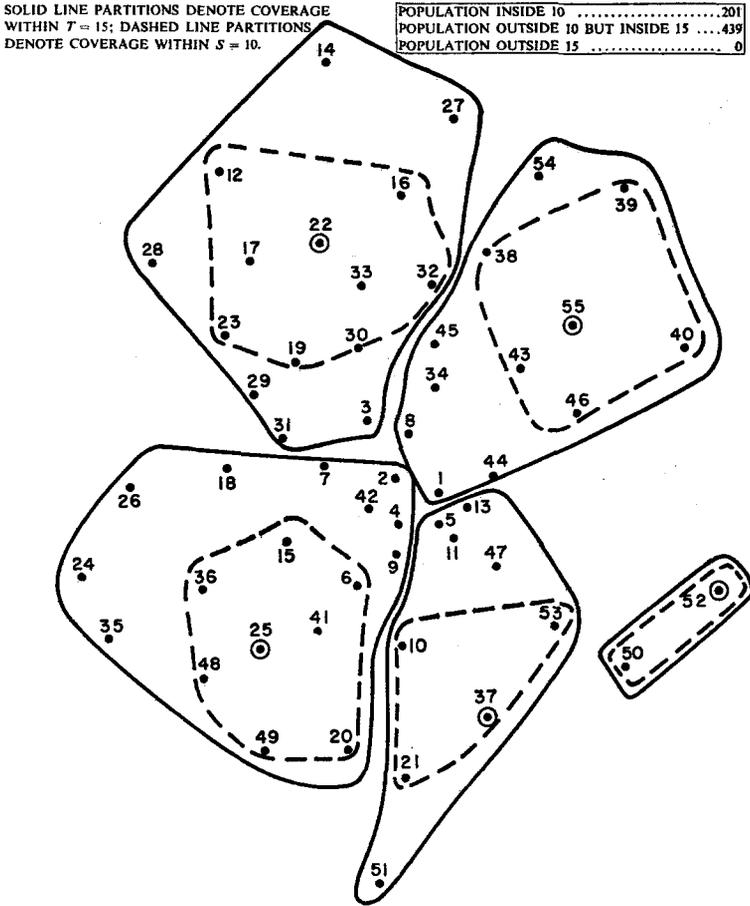
[22] Toregas and ReVelle, *op. cit.*

FIGURE 2. A LOCATION SET COVERING SOLUTION FOR $S = 15$
(DATA FROM SWAIN, REF. 20)

located at node 25, indicating that node 42 lies at a distance greater than 10 from its closest facility, node 25. Figure 2 also shows, for this particular location set covering solution, that 201 of the total population are covered within a distance of 10; the rest of the population, 439, is located between 10 and the maximal service distance of 15.

It was stated in the previous section that the maximal covering location problem with mandatory closeness constraints could be used to choose among the alternative optima of the location set covering problem, the solution which achieves the best coverage for some more desirable "inner" distance. For example, by solving the maximal covering location problem with mandatory closeness constraints with $S = 10$ and $T = 15$, still using five facilities we could determine the solution that provides the best coverage within $S = 10$ while maintaining com-
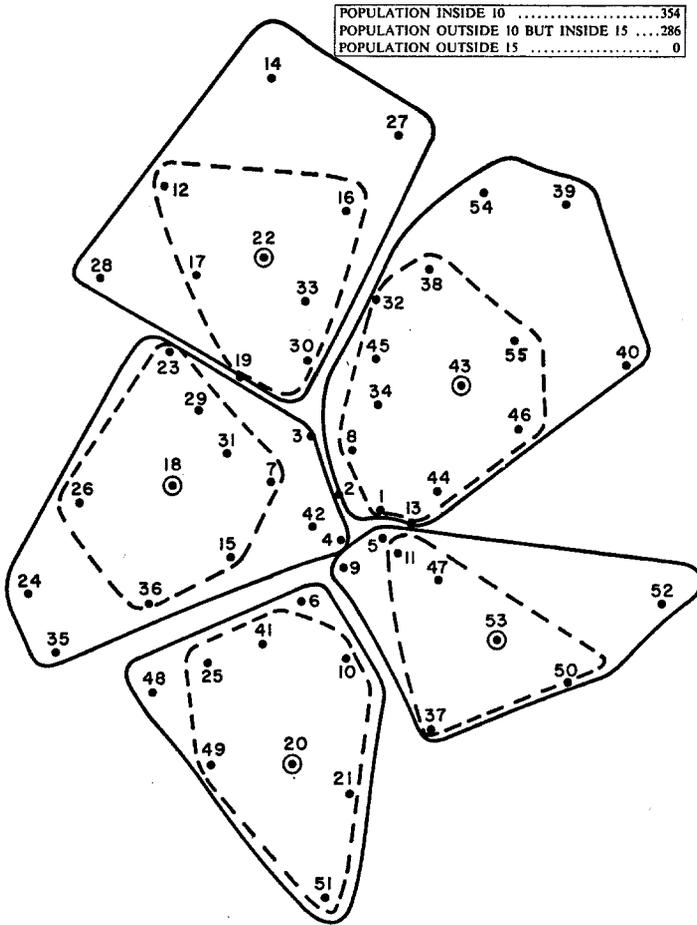
FIGURE 3. AN OPTIMAL SOLUTION TO THE MAXIMAL COVERING LOCATION
PROBLEM WITH MANDATORY CLOSENESS CONSTRAINTS
(DATA FROM SWAIN, REF. 20)

plete coverage within $T = 15$.

It is entirely possible that the decision maker may feel it desirable to cover as
much population as possible within 10 if he can still provide complete coverage
within 15.   After all, he appears to be giving up nothing.   The maximal covering
location problem with mandatory closeness constraints with $S = 10$ and $T = 15$
was solved for this network and the solution is presented in Figure 3.   Essentially,
Figure 3 depicts the alternative optimum of the location set covering problem for
$S = 15$ which covers the maximum population within 10.   Notice that the coverage
within 10 for this solution amounts to 354 out of 640.   By the nature of the maximal
covering problem with mandatory closeness constraints, it is possible to state
that no more than 354 can be covered within 10 while maintaining complete coverage

within 15, given no more than five facilities. The solution in Figure 3 is clearly a better solution than the one given in Figure 2, for the coverage within 10 is 75 percent better than the solution given in Figure 2. Therefore, by using the maximal covering location problem with mandatory closeness constraints, it is possible to easily determine the most desirable location set covering solutions. In the above problem, the use of the maximal covering location approach with mandatory closeness constraints has provided substantial improvement over the original location set covering solution given in Figure 2.

We know that it is impossible to cover more than 354 within 10 while maintaining complete coverage within 15 using five facilities. Thus, it would be futile to try to cover more than 354 within 10 unless either more facilities were allowed or the mandatory closeness constraints of 15 were relinquished.
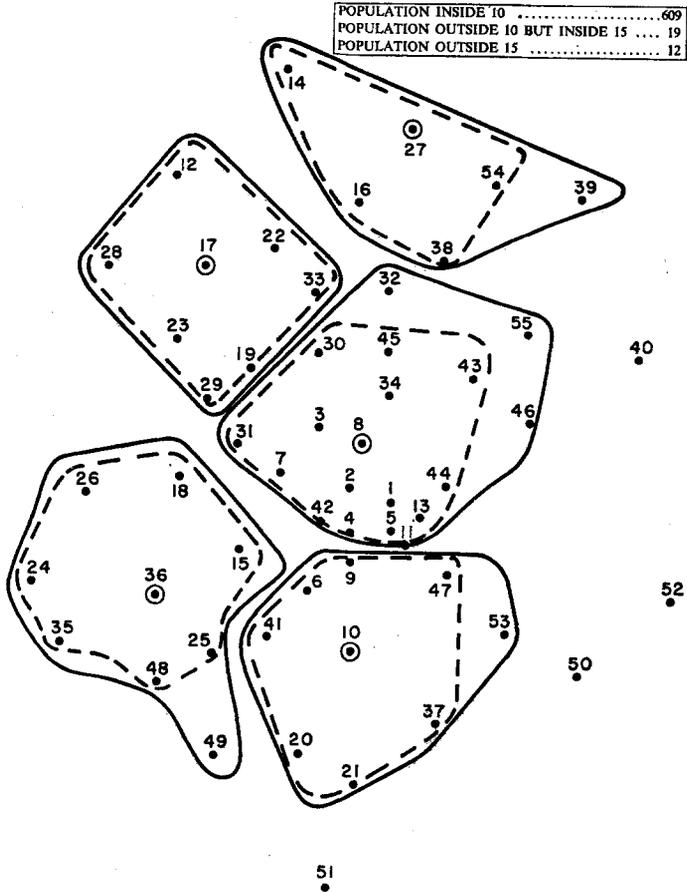

FIGURE 4. AN OPTIMAL SOLUTION TO THE MAXIMAL COVERING LOCATION PROBLEM FOR $S = 10$ (DATA FROM SWAIN, REF. 20)

In the above example problem, it is, in fact, possible to achieve a very large "maximal cover" within $S = 10$ using five facilities if one relaxes the mandatory closeness constraint of 15. From the viewpoint of the decision maker, achieving maximal coverage by five facilities within a distance of 10 could be very desirable, if the resultant coverage within 10 is sufficiently improved from 354 and if the number of people outside of 15 is not excessive.

Figure 4 gives a maximal covering solution for $S = 10$. A node in the graph which is not covered within $T = 15$ is not included in any partition. Notice that the population covered within $S = 10$ has reached 609 out of a total population of 640. In this case, only four nodes with a combined population of 12 are at a distance greater than 15 from their closest facilities.

In order to cover all demand nodes within 15, the amount covered within 10 must decrease from 609, given by the maximal covering problem with $S = 10$, to at most 354 which value was given by the maximal covering problem with a mandatory closeness constraint of 15. This reduction of 255 in the population covered within 10 can be viewed as one of the costs of maintaining *all* nodes within a maximum distance of 15. That is, the mandatory closeness constraints ensured that a population totaling 12 out of 640 (at four out of 55 nodes) found service within 15 distance units. At the same time, a population totaling 255 was forced outside the distance of 10.

In summary, it has been shown that the maximal covering location problem and the maximal covering location problem with mandatory closeness constraints are powerful location techniques which build on and, in fact, supplant the location set covering problem. The maximal covering location problem with mandatory closeness, furthermore, may be used to provide the most desirable of the solutions to the location set covering problem. It is also easy to see that "good" locational decisions can be made by all three techniques, but enlightened use of the maximal covering location problems appears to lead to superior patterns of population coverage. The above example problems show that the additional information gained by using these two techniques in conjunction with the location set covering approach can lead to better decision making with regard to public facility location.